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## Feedback coupling and self-organization

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**Abstract.** The correlation function of a kinetic Ising model offers a crossover from an exponential to an algebraic decay with a universal exponent when a feedback coupling is introduced via a stochastic field. The correlation of this field is directly related to the current spin correlation function. Due to the feedback coupling, at low temperatures there appears a strong slowing down resulting in a self-similar behaviour. We argue that such a self-organized mechanism should be relevant in glasses.

Based on a simple kinetic Ising model we demonstrate in an analytical manner the crossover from an exponential to an algebraic decay of the correlation function. This study is motivated by the hypothesis proposed by Bak *et al* [1] that systems consisting of many interacting constituents may exhibit some general characteristic behaviour. In particular, such typical behaviour develops without any significant tuning of the system from outside. The state into which the systems organize themselves is characterized by the lack of any typical time or length scale manifested by an algebraic decay of the correlation function. In spite of a great effort in studying self-organized criticality (SOC) the precise significance is still controversial [2–4]. Originally, SOC was introduced as a general scope to consider fractals and  $1/f$  noise as a result of the dynamical evolution of extended systems. The idea behind the phenomenon had been illustrated by computer simulations. A slow driving force leads to a stationary state characterized by a distribution of amplitudes [1]. More refined techniques such as the renormalization group approach are applied to different problems, such as recently to the directed sandpile models [4]. It seems that all the experiments share with SOC models the slow external driving and the avalanche response.

Here, we consider a simple kinetic Ising model with a stochastic driving force. As the new feature of the model this driving force is related via a feedback coupling to the current-time-dependent spin–spin correlation function of the system itself. That feedback mechanism leads immediately to a crossover from an exponential decay in the initial time regime to an algebraic decay of the correlation function in the long time limit for low temperatures. Thus, the feedback gives rise to SOC. Note that recently stochastic feedback and the regulation of biological rhythms has been considered [5].

Another motivation is the slow dynamics inevitably related to glasses. Although the transition from a liquid to a supercooled liquid and further to a glass is studied with different methods it remains one of the unsolved challenging problems in the theory of phase transitions [6]. The main reason for this is that such systems reveal on short time scales and on small spatial scales, a solid-like behaviour, whereas in the long time limit and on large spatial scales the system behaves like a liquid. Therefore, such a liquid–glass transition is accomplished

by a very pronounced slowing down of the relaxation time. The relaxation patterns are non-exponential in time and depend strongly on temperature. The slow dynamics is also a feature of conventional phase transitions [7]; however, frozen liquids do not evolve an observable long-range correlated or an ordered state which is persistent in time. Obviously the liquid–glass transition is dynamic in origin characterized by a high cooperativity of local processes [8].

In order to achieve progress within an analytical approach, there are several possibilities, the foremost being the use of mode-coupling theory [6]. Recently, we have studied a glassy system [9] in the frame of a spin facilitated model due to Fredrickson and Andersen, see [10, 11], compare also [12–16]. By mapping the model onto a kinetic Ising model with local hopping rates depending on the configuration in the neighbourhood, we obtained the characteristic slowing down. To make the relation to a kinetic Ising model more transparent, the supercooled liquid should be divided in small cells, say of nanometre scale, which are characterized by different local mobilities. As a coarse-grained variable, let us introduce a lattice gas representation. If the lattice cell  $i$  is occupied by mobile particles the state is assigned to  $n_i = 1$ , whereas in the case of an immobile state we set  $n_i = 0$ . As usual the occupation number can be related to a spin variable by  $S_i = 1 - 2n_i$ . A certain configuration is denoted by a state vector  $\vec{n} = \{n_1, n_2, \dots\}$ . Furthermore, assuming that the dynamics is based on hopping processes, there occurs a cooperative rearrangement of certain cells in order to change more immobile cells into more mobile ones and *vice versa*. The cooperativity leading to the ultra-slow behaviour originates from the fact that a given cell is embedded in an environment of other cells of different mobilities, which can be modelled by assuming that the flip rates depend on the neighbouring configuration. This kinetic confinement yields the slowing down of the correlation. In particular, a non-Arrhenius behaviour in the relaxation time is observed [9, 15]. A further approach consists of the feedback coupling introduced later. Note that a real liquid cannot be mapped completely onto an Ising model. Nevertheless, the modified kinetic Ising model reflects the main features of glassy systems and should be considered as a reasonable approach to describing the characteristic slowing down of supercooled liquid dynamics. Furthermore, the model can be extended in a straightforward manner including other degrees of freedom for instance vacancies.

The dynamics is introduced via the master equation written in the symbolic form

$$\partial_t P(\vec{n}, t) = L' P(\vec{n}, t). \quad (1)$$

Here  $P$  is the probability that the configuration  $\vec{n}$  is realized at time  $t$ . The evolution operator  $L'$  will be specified later. Furthermore, let us introduce annihilation and creation operators to formulate a state in terms of occupation number operators, where in the present paper those operators have the eigenvalues 0 and 1. Thus, the problem is to formulate the dynamics in such a way that this restriction is taken into account [17–21], for recent reviews see [22, 23]. The situation in mind can be analysed in a seemingly compact form using the master equation [19, 24, 25] introduced above (1). Following [19, 21, 24–26] the probability distribution  $P(\vec{n}, t)$  is related to a state vector  $|F(t)\rangle$  in a Fock space according to  $P(\vec{n}, t) = \langle \vec{n} | F(t) \rangle$  with the basis vectors  $|\vec{n}\rangle$  composed of second quantized operators. The master equation (1) can be transformed to an equivalent equation in a Fock space

$$\partial_t |F(t)\rangle = \hat{L} |F(t)\rangle. \quad (2)$$

The operator  $L'$  in (1) is mapped onto the operator  $\hat{L}$ . Up to now the procedure is independent of the operators used. Originally, the method was applied for the Bose case [24–26]. An extension to restricted occupation numbers (two discrete orientations) has been proposed [17–21] introducing Pauli operators. These operators commute at different cells and anti-commute at the same lattice cell. A further extension to a  $p$ -fold occupation number is possible [27].

The relation between the quantum-like formalism and the probability approach is given by

$$|F(t)\rangle = \sum_{n_i} P(\vec{n}, t) |\vec{n}\rangle. \quad (3)$$

As was first shown by Doi [24] the average of an arbitrary physical quantity  $B(\vec{n})$  is defined by the average of the corresponding operator  $B(t)$

$$\langle B(t) \rangle = \sum_{n_i} P(\vec{n}, t) B(\vec{n}) = \langle s | B | F(t) \rangle \quad (4)$$

with the state function  $\langle s | = \sum \langle \vec{n} |$ . Using the relation  $\langle s | \hat{L} = 0$ , the evolution equation for an operator  $A$  can be written as

$$\partial_t \langle A \rangle = \langle s | [A, \hat{L}] | F(t) \rangle. \quad (5)$$

It seems necessary to note that all the dynamical equations covering the classical problem are determined by the commutation rules of the underlying operators and the structure of the evolution operator  $\hat{L}$ . In our case the dynamics of the model is given by spin-flip processes indicating a change of the local mobilities or local densities.

The method can be extended by including the mutual interaction between different cells and by considering finite temperatures. To this aim we write the evolution operator for local flip processes in the form [28, 29]

$$\begin{aligned} \hat{L} = \nu \sum_i [(1 - d_i) \exp(-\beta H/2) d_i^\dagger \exp(\beta H/2)] \\ + [(1 - d_i^\dagger) \exp(-\beta H/2) d_i \exp(\beta H/2)]. \end{aligned} \quad (6)$$

Here,  $\nu$  is a hopping rate,  $\beta = T^{-1}$  is the inverse temperature of the heat bath and  $H$  is the Hamiltonian which is assumed to be

$$H = \sum_i h_i(t) n_i. \quad (7)$$

The local field  $h_i(t)$  is composed of static and stochastic parts

$$h_i(t) = h + \xi_i(t) \quad (8)$$

with mean value  $\bar{\xi}_i = 0$ . A positive static part  $h > 0$  favours the solid state because  $h$  can be considered as the energy difference per cell between the liquid and the solid state

$$h = \frac{E_L - E_S}{N}.$$

Using the algebraic properties of the Pauli operators  $d_i$  and  $d_i^\dagger$  we get

$$\hat{L} = \nu \sum_i [(1 - d_i) d_i^\dagger \exp(-\beta h_i/2) + (1 - d_i^\dagger) d_i \exp(\beta h_i/2)]. \quad (9)$$

The first term describes the change of the mobility from a solid-like to liquid-like state. The second term characterizes the transition from the more mobile (liquid) state to a more immobile (solid) state. In the case of a constant field  $h_i = h$  the hopping rate for the first process is proportional to  $\exp(-h/2T)$ , indicating that the transition to a liquid-like state for low temperatures is not very probable. In contrast, the transition to a solid-like state is governed by the second term with a rate proportional to  $\exp(h/2T)$ , which favours the occurrence of solid regions for low temperatures.

The correlation function is defined by

$$C_i(t) = \langle n_i(t) n_i(0) \rangle. \quad (10)$$

Based on the interpretation of the  $n_i$  introduced above, this function characterizes the liquid state. Using equation (5) the correlation function fulfils, for a given realization of the stochastic field  $\xi_i(t)$ , the evolution equation

$$v^{-1} \partial_t C_i(t) = -2 \cosh\left(\frac{h_i(t)}{2T}\right) C_i(t) + \exp\left(-\frac{h_i(t)}{2T}\right). \tag{11}$$

Obviously, the correlation functions offer an exponential decay when the stochastic field is absent. Because a more pronounced critical slowing down is expected for low temperatures we analyse the correlation function in the limit  $T \rightarrow 0$ . For this case we only take into account exponential increasing terms in equation (11). The complete time-dependent correlation function follows after performing the average over the random field  $\xi_i(t)$

$$C(t) \equiv \overline{C_i(t)} = C_i(0) \overline{\exp(-v F_i(t))}$$

$$\text{with } F_i(t) = \exp(h/2T) \int_0^t dt' \exp(\xi_i(t')/2T). \tag{12}$$

Now let us introduce a feedback coupling by assuming that the stochastic field is coupled directly to the current correlation function. In other words, we suppose

$$h_i(t) = h + \xi(t) \quad \bar{\xi}_i = 0 \quad \overline{\xi_i(t)\xi_i(0)} = \kappa C(t) \tag{13}$$

where the coupling parameter  $\kappa$  is of the order  $O(1)$ . Performing the average over the stochastic field, we obtain in a first-order cummulant expansion

$$v \overline{F_i(t)} \equiv F(t) = \frac{t}{\tau} \quad \text{with} \quad \tau = \exp\left[-\frac{h}{2T} - \frac{\kappa C(0)}{8T^2}\right] \equiv \tau_0 \exp\left(-\frac{\kappa C_0}{8T^2}\right). \tag{14}$$

Here,  $\tau_0$  is the relaxation time of the conventional kinetic Ising model for low temperatures. The second-order cummulant expansion leads to

$$\frac{1}{2}[\overline{F_i^2} - \overline{F_i}^2] = \frac{R(T)}{v^2} I(t; T)$$

$$\text{with } I(t; T) = \int_0^t dt_1 \int_0^{t_1} dt_2 (e^{\kappa C(t_1-t_2)/4T^2} - 1)$$

$$\text{and } R(T) = \frac{v^2}{2} e^{[h/T + \kappa C_0/(4T^2)]}. \tag{15}$$

The averaged correlation function reads

$$C(t) = C_0 \exp\left[-\frac{t}{\tau} + R(T)I(t; T)\right]. \tag{16}$$

To find out a solution of this self-consistent equation for  $C(t)$  we look for the corresponding differential equation. Without any further approximation, we obtain the following equation where the dots mean the time derivatives

$$C(t)\ddot{C}(t) = \dot{C}(t)^2 + R(T)C(t)^2 \left[ \exp\frac{\kappa C(t)}{4T^2} + \exp\frac{\kappa C(-t)}{4T^2} - 2 \right]. \tag{17}$$

A solution can be obtained by assuming

$$C(t_1 - t_2)/4T^2 \ll 1 \quad R = \frac{1}{4(T\tau)^2}.$$

The validity of this approximation can be checked at the final result. Equation (17) is simplified to

$$C\ddot{C} = \dot{C}^2 + bC^3 \quad \text{with } b = \frac{\kappa}{4} \frac{1}{(T\tau)^2}. \tag{18}$$

Obviously, this equation exhibits a power-law solution. Making the ansatz  $C(t) = At^{-\alpha}$ , we obtain the asymptotic solution

$$C(t) = \frac{8}{\kappa} \left( \frac{T\tau}{t} \right)^2. \quad (19)$$

To find out corrections to this behaviour we set

$$C(t) = t^{-2} f \left( \frac{T\tau}{t} \right)$$

yielding a differential equation for  $r(z) \equiv \ln(bf(z)/2)$

$$-z^2 r''(z) + 2(e^{r(z)} - 1) = 0 \quad \text{with } z = \frac{T\tau}{t}. \quad (20)$$

Comparing the asymptotic solution equation (19) with the last equation, we conclude that  $\lim_{z \rightarrow \infty} r(z) \rightarrow 0$ . Thus, equation (20) should be solved for small  $r(z)$ . This results in

$$C(t; T) \simeq \frac{8(T\tau)^2}{\kappa t^2} \exp \left[ \frac{cT\tau}{t} \right] \quad (21)$$

where  $c$  is an arbitrary constant. For large time the spin (or density) correlation function  $C(t/T)$  decays algebraically with the universal exponent 2. Otherwise, for a fixed time the correlation function is also an algebraic one as a function of temperature. Obviously,  $C(t; T)$  in (21) depends on the ratio between time and temperature. This reflects the common believe for glasses that low temperatures and a large time scale are related intimately. The algebraic slowing down of  $C(t; T)$  is self-organized by the feedback mechanism.

In the present paper we have explicitly demonstrated that a feedback coupling can be considered as a new mechanism giving rise to a crossover from an exponential to an algebraic decay. Due to the feedback coupling the system is forced to be self-similar with respect to the asymptotic behaviour of the time correlation function. Obviously, such a behaviour is characteristic for the freezing process in glasses where the density correlation function for the liquid state should decrease with increasing time and with decreasing temperature.

It would be interesting to study other more complex systems to test whether they also reveal a self-organized behaviour caused by a feedback coupling.

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